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ON THE BEHAVIOUR OF SOLUTIONS OF DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

The behaviour of singular solution $u(x)$ of quasilinear elliptic equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x}\right) - |u|^{q-2}u = 0, \quad x \in B_1(0) \setminus \{0\} \tag{1}$$

is studied. We assume Caratheodory's conditions for coefficients $a_i(x,\xi)$ and inequalities

$$\sum_{i=1}^{n} a_i(x,\xi)\xi_i \geq \nu_1 |x|^{\sigma}|\xi|^p, \quad |a_i(x,\xi)| \leq \nu_2 |x|^{\sigma}|\xi|^{p-1}$$

with positive constants ν_1, ν_2, $\sigma \in (p-n, n(p-1))$, $1 < p < n$.

Following results are established:

1) the boundedness of the solution $u(x)$ of the equation (1) under conditions $q > 1$,

$$|u(x)| \leq M_1 |x|^{-\mathcal{P}+\delta} \quad for \quad 0 < |x| \leq \frac{1}{2}, \ \mathcal{P} = \frac{n-p+\sigma}{p-1}, \ \delta > 0;$$

2) the estimate

$$|u(x)| \leq M_3 |x|^{-Q} \quad for \quad 0 < |x| \leq \frac{1}{2}, \ Q = \frac{p-\sigma}{q-p}, \ q > p;$$

3) the removability of each isolated singularity of the solution of the equation (1) if $Q < P$.

These assertions are generalized well-known results of V.A.Kondratiev and E.M.Landis established for $\sigma = 0$ for the equation (1) with $p = 2$ and linear principal part.

1. We study the behaviour of a singular solution $u(x)$ of the following quasilinear elliptic equation:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x}\right) - |u|^{q-2}u = 0, \quad x \in \Omega = B_1(0) \setminus \{0\} \tag{1}$$

where $B = B_1(0)$ is the ball of radius 1 with a center 0, and q is a real positive number precised later.

We assume that the coefficients $a_i(x,\xi)$ are Caratheodory's functions and that they satisfy the following ellipticity condition:

$$\sum_{i=1}^{n} a_i(x,\xi)\xi_i \geq \nu_1 |x|^{\sigma}|\xi|^p \tag{2}$$

and the growth condition:

$$|a_i(x,\xi)| \leq \nu_2 |x|^{\sigma}|\xi|^{p-1} \tag{3}$$

for $(x,\xi) \in \Omega \times \mathbb{R}^n$ and with positive constants ν_1, ν_2 and numbers σ, p such that

$$p - n < \sigma < n(p-1), 1 < p < n. \tag{4}$$

A function $u(x) \in W^{1,p}_{loc}(\Omega) \cap L^q_{loc}(\Omega)$ is called a solution of the equation (1) in $B \setminus \{0\}$ if for an arbitrary function $\varphi \in W^{1,p}(B) \cap L^q(B)$ that is equal to zero near $\partial B \cup \{0\}$ we have

$$\int\limits_\Omega \sum_{i=1}^n a_i(x, \frac{\partial u}{\partial x}) \frac{\partial \varphi}{\partial x_i} dx + \int\limits_\Omega |u|^{q-2} u \varphi dx = 0. \tag{5}$$

We will say that the solution $u(x)$ of the equation (1) in $B \setminus \{0\}$ has at $\{0\}$ removable singularity if the integral identity (5) is true for an arbitrary function $\varphi \in W^{1,p}(B) \cap L^q(B)$ that is equal to zero near ∂B.

Main Results of this paper are following Theorems.

THEOREM 1. *Let $u(x)$ be a solution of the equation (1) in $B \setminus \{0\}$. Assume that $q > 1$, the inequalities (2), (3) and the estimate*

$$|u(x)| \le M_1 |x|^{-P+\delta}, \quad P = \frac{n-p+\sigma}{p-1} \tag{6}$$

are satisfied for $0 < |x| \le \frac{1}{2}$ with some positive numbers M_1, δ. Then there exists a positive constant M_2 such that the estimate

$$|u(x)| \le M_2 \quad 0 < |x| \le \frac{1}{2} \tag{7}$$

holds.

THEOREM 2. *Assume that conditions (2), (3) are satisfied and $q > p$. Let $u(x)$ be a solution of the equation (1) in $B \setminus \{0\}$. Then the estimate*

$$|u(x)| \le M_3 |x|^{-Q}, \quad 0 < |x| \le \frac{1}{2} \tag{8}$$

holds with $Q = \frac{p-\sigma}{q-p}$ and some positive constant M_3.

THEOREM 3. *Assume that conditions (2), (3) are satisfied and $q > \frac{np-p+\sigma}{n-p+\sigma}$. Then for an arbitrary solution $u(x)$ in $B \setminus \{0\}$ the singularity at $\{0\}$ is removable.*

Note that V.A. Kondratiev and E.M. Landis in [1] established analogous result for linear equation of type (1), that corresponds to $p = 2, \sigma = 0$.

2. Proof of the Theorem 1. Let us substitute in (5) a test function

$$\varphi = (1 + |u|)^{-\alpha} u \, \psi^p(x) \, \eta_r^p(x) \quad , 0 < \alpha < 1,$$

where $\psi(x) = 1$ in $B_{\frac{1}{2}}(0), 0 \le \psi(x) \le 1$, $\psi(x) = 0$ outside $B_{\frac{3}{4}}(0)$, $\left|\frac{\partial \psi}{\partial x}\right| \le c_0$, $\eta_r(x) = 1$ outside $B_{2r}(0)$, $0 \le \eta_r(x) \le 1$, $\eta_r(x) = 0$ inside $B_r(0)$, $\left|\frac{\partial \eta_r}{\partial x}\right| \le \frac{c_0}{r}$, where r is enough small.

Using inequalities (2), (3), Young inequality and simple calculations we obtain the estimate

$$\begin{aligned} &\int\limits_B (1 + |u|)^{-\alpha} \left|\frac{\partial u}{\partial x}\right|^p \psi^p(x) \eta_r^p(x) |x|^\sigma dx \le \\ &\le C_1 \int\limits_{B \setminus B_r(0)} (1 + |u|)^{-\alpha} |x|^\sigma |u|^p \left(\left|\frac{\partial \psi}{\partial x}\right|^p + \left|\frac{\partial \eta_r}{\partial x}\right|^p \right) dx. \end{aligned} \tag{9}$$

The constant C_1 here and other constants C_i in the proof of the Theorem 1 depend only on $\nu_1, \nu_2, n, p, \sigma, M_1$.

Now we want to pass to the limit when $r \to 0$. The term of sum in right hand side of (9) with a derivative of ψ is estimated as following

$$\int\limits_{B \setminus B_r(0)} (1+|u|)^{-\alpha}|u|^p|x|^\sigma \left|\frac{\partial \psi}{\partial x}\right|^p dx \leq C_2 \left\{1 + r^{n+\sigma-\left(\frac{n-p+\sigma}{p-1}-\delta\right)(p-\alpha)}\right\}. \quad (10)$$

The right-hand side of (10) is bounded for $r \to 0$ if

$$\alpha > \frac{n+\sigma-[p+\delta(p-1)]p}{n+\sigma-[p+\delta(p-1)]}. \quad (11)$$

The second term of the right-hand side of (9) has the estimate

$$\int\limits_{B \setminus B_r(0)} (1+|u|)^{-\alpha}|u|^p \left|\frac{\partial \eta_r}{\partial x}\right|^p |x|^\sigma dx \leq C_3\, r^{n+\sigma-\left(\frac{n-p+\sigma}{p-1}-\delta\right)(p-\alpha)-p}. \quad (12)$$

The last expression will go to zero when $r \to 0$ if

$$n+\sigma-\left(\frac{n-p+\sigma}{p-1}-\delta\right)(p-\alpha)-p > 0,$$

this implies the condition on α:

$$\alpha > \frac{n+\sigma-p-\delta p(p-1)}{\sigma+n-p-\delta(p-1)}. \quad (13)$$

We choose α as in (13) with $\alpha < 1$. By monotone convergence theorem applied to the inequality (9) we obtain:

$$\int\limits_B (1+|u|)^{-\alpha}\left|\frac{\partial u}{\partial x}\right|^p \psi^p(x)|x|^\sigma dx \leq C_4 \int\limits_B (1+|u|)^{-\alpha}|u|^p\left|\frac{\partial \psi}{\partial x}\right|^p |x|^\sigma dx. \quad (14)$$

Now we want to prove boundedness of $u(x)$ in B. We shall use Moser iteration process. We substitute in (5)

$$\varphi = (1+|u|_k)^l(1+|u|)^{-\alpha}u\psi^s(x)\eta_r^s(x) \quad , s>0, l \geq 0$$

where $|u|_k = \min\{|u(x)|, k\}$, $k>0$, $\psi(x), \eta_r(x)$ are the same functions as before.

After standard calculations we have the inequality

$$\int\limits_B (1+|u|_k)^l(1+|u|)^{-\alpha}\left|\frac{\partial u}{\partial x}\right|^p \psi^s(x)\eta_r^s(x)|x|^\sigma dx \leq$$
$$\leq C_5\, s^p \int\limits_B (1+|u|)^{-\alpha}|u|^p(1+|u|_k)^l\psi^{s-p}(x)\eta_r^{s-p}(x)\cdot \quad (15)$$
$$\cdot\left(\left|\frac{\partial \psi}{\partial x}\right|^p + \left|\frac{\partial \eta_r}{\partial x}\right|^p\right)|x|^\sigma dx.$$

We pass to the limit in (15) if $r \to 0$ and obtain

$$\int_B (1+|u|_k)^l (1+|u|)^{-\alpha} \left|\frac{\partial u}{\partial x}\right|^p \psi^s(x)|x|^\sigma dx \le \\ \le C_5\, s^p \int_B (1+|u|_k)^l (1+|u|)^{-\alpha}(1+|u|)^p \psi^{s-p}(x)|x|^\sigma dx. \tag{16}$$

We will apply Moser method. We shall use embedding theorem

$$W_p^1(\Omega, |x|^\sigma) \subset L_{p\kappa}(\Omega, |x|^\sigma) \ \text{ with } \ 1 < \kappa < \frac{\sigma+n}{\sigma+n-p}.$$

We apply the embedding theorem to the right hand side of (16) and we obtain

$$\int_B (1+|u|_k)^l (1+|u|)^{p-\alpha}\psi^{s-p}(x)|x|^\sigma dx \le \\ \le C_6\, l^\kappa \left\{ \int_B (1+|u|_k)^{\frac{l}{\kappa}+(\alpha-p)(1-\frac{1}{\kappa})} \cdot \\ \cdot (1+|u|)^{p-\alpha}\psi^{\frac{s-p}{\kappa}-p}(x)dx \right\}^\kappa. \tag{17}$$

Now we can apply Moser method, which gives the inequality

$$\max_{x\in B_{\frac{1}{2}}} (1+|u(x)|_k)^{l_0} \le C_7 \int_B (1+|u|_k)^{l_0}(1+|u|)^{p-\alpha}|x|^\sigma \psi^{s_0-p}(x)dx \tag{18}$$

with numbers l_0, s_0 such that $l_0 > 0, s_0 \ge p$.

We need to choose l_0 such that

$$\int_B (1+|u|)^{p-\alpha+l_0}|x|^\sigma dx \le C_8 \tag{19}$$

or, equivalently we must choose l_0 from the condition

$$n+\sigma - \left(\frac{n-p+\sigma}{p-1} - \delta\right)(p-\alpha+l_0) > 0.$$

So we can choose positive l_0 if α satisfies the inequality (13). Now inequalities (18), (19) imply a boundedness of $u(x)$ in $B_{\frac{1}{2}}$. This is the end of the proof of the theorem.

3. Proof of the Theorem 2. We introduce a function $\xi_\rho(x)$ such that $\xi_\rho(x) = 1$ for $\frac{\rho}{2} < |x| < \rho$, $\xi_\rho(x) = 0$ outside of $\left\{\frac{\rho}{4} \le |x| \le \frac{5\rho}{4}\right\}$ and $\left|\frac{\partial \xi_\rho(x)}{\partial x}\right| \le \frac{c_9}{\rho}$.

We substitute in (5) a test function

$$\varphi = |u(x)|^{r-1}u(x)\ \xi_\rho^s(x) \qquad r \ge 1,\ s \ge p.$$

Standard calculations lead to following estimate

$$\frac{r\nu_1}{2}\int\limits_{\Omega}|x|^\sigma|\nabla u|^p|u(x)|^{r-1}\xi_\rho^s(x)dx+\int\limits_{\Omega}|u(x)|^{q+r-1}\xi_\rho^s(x)dx\leq$$
$$\leq C_{10}\int\limits_{\Omega}\left(\frac{s}{r}\right)^p|x|^\sigma|u(x)|^{r+p-1}\frac{1}{\rho^p}\xi_\rho^{s-p}(x)dx. \tag{20}$$

We recall now the weighted Sobolev embedding theorem (see [4])

$$\left(\frac{1}{\rho^{\sigma+n}}\int\limits_{B_\rho(x_o)}|\phi|^{\kappa p}|x|^\sigma dx\right)^{\frac{1}{\kappa p}}\leq C_{11}\rho\left(\frac{1}{\rho^{\sigma+n}}\int\limits_{B_\rho(x_0)}|\nabla\phi|^p|x|^\sigma dx\right)^{\frac{1}{p}}. \tag{21}$$

We apply this last inequality to the function $\phi=|u(x)|^{\frac{r+p-1}{\kappa p}}\xi_\rho^{\frac{s-p}{\kappa p}}(x)$

$$\int\limits_{\Omega}\left[u^{\frac{r+p-1}{\kappa p}}(x)\ \xi_\rho^{\frac{s-p}{\kappa p}}(x)\right]^{\kappa p}|x|^\sigma dx\leq\frac{C_{12}(r+s+p)^{\kappa p}}{\rho^{(\sigma+n)(\kappa-1)}}\rho^{\kappa p}.$$
$$\cdot\left[\int\limits_{\Omega}|u(x)|^{\frac{r+p-1}{\kappa}-p}|\nabla u|^p\xi_\rho^{\frac{s-p}{\kappa}}(x)|x|^\sigma+|u(x)|^{\frac{r+p-1}{\kappa}}\xi_\rho^{\frac{s-p}{\kappa}-p}(x)\left(\frac{c}{\rho}\right)^p|x|^\sigma dx\right]^\kappa \tag{22}$$

Using the inequality (20) we get the estimate

$$\frac{1}{\rho^{\sigma+n}}\int\limits_{\Omega}|x|^\sigma\left(|u(x)|^{\frac{r+p-1}{\kappa p}}\xi_\rho^{\frac{s-p}{\kappa p}}(x)\right)^{\kappa p}dx\leq$$
$$\leq\frac{C_{13}(r+s+p)^{2\kappa p}}{r^{\kappa p}}\left(\frac{1}{\rho^{\sigma+n+p}}\int\limits_{\Omega}|u(x)|^{\frac{r+p-1}{\kappa}}\xi_\rho^{\frac{s-p}{\kappa}-p}(x)|x|^\sigma dx\right)^\kappa. \tag{23}$$

Denote $r_i=\kappa^i p+1-p,\ s_i=\left(\frac{pq}{q-p}-\frac{p}{1-\kappa}\right)\kappa^i+\frac{p}{1-\kappa},$

$$I_i=\frac{1}{\rho^{\sigma+n}}\int\limits_{\Omega}|x|^\sigma|u(x)|^{r_i+p-1}\xi_\rho^{s_i-p}(x)dx.$$

Using these notations we can write the inequality (23) with $r=r_i,\ s=s_i$ in following way

$$I_i\leq C_{14}\,\kappa^{2\kappa p i}I_{i-1}^\kappa. \tag{24}$$

Iterating this inequality we obtain

$$\max_{\frac{\rho}{2}<|x|<\rho}|u(x)|^p\leq\frac{C_{15}}{\rho^{\sigma+n}}\int\limits_{\Omega}|x|^\sigma|u(x)|^p\xi_\rho^{s_0-p}(x)dx. \tag{25}$$

Now we use (20)(if $r=r_0=1$ and $s=s_0$) and we find

$$\int\limits_{\Omega}u^q(x)\xi_\rho^{s_0}(x)dx\leq\frac{C_{16}}{\rho^p}\int\limits_{\Omega}|x|^\sigma|u(x)|^p\xi_\rho^{s_0-p}(x)dx. \tag{26}$$

So, by using Young's inequality and (26), we have

$$\int_\Omega |x|^\sigma |u(x)|^p \xi_\rho^{s_0-p}(x)dx = \int_\Omega \rho^{\frac{p^2}{q}} |u(x)|^p \rho^{-\frac{p^2}{q}} [\xi_\rho(x)]^{-p} |x|^\sigma \xi_\rho^{s_0}(x)dx \le$$

$$\le C_{17}\,\varepsilon^{\frac{q}{p}} \int_\Omega |x|^\sigma |u(x)|^p \xi_\rho^{s_0-p}(x)dx + C_{18}\,\varepsilon^{-\frac{q}{q-p}} \int_\Omega \rho^{-\frac{p^2}{q-p}} |x|^{\frac{\sigma q}{q-p}} \xi_\rho^{s_0-\frac{pq}{q-p}}(x)dx,$$

so

$$\int_\Omega |x|^\sigma |u(x)|^p\, \xi_\rho^{s_0-p}(x)dx \le C_{19}\,\rho^{\frac{-p^2}{q-p}+n+\frac{\sigma q}{q-p}}$$

and then

$$\max_{\frac{\rho}{2}<|x|<\rho} |u(x)|^p \le \frac{C_{20}}{\rho^{\sigma+\frac{p^2}{q-p}-\frac{\sigma q}{q-p}}}.$$

In the end we have

$$\max_{\frac{\rho}{2}<|x|<\rho} |u(x)| \le \frac{C_{21}}{\rho^{\frac{p-\sigma}{q-p}}}$$

and the proof of the Theorem 2 is completed.

4. Proof of the Theorem 3. Theorems 1, 2 imply that an arbitrary solution of the equation (1) in $B\setminus\{0\}$ is bounded in $B_{\frac{1}{2}}$ if $Q<P$. It means the boundedness of the solution $u(x)$ in $B_{\frac{1}{2}}$ for $q>\frac{np-p+\sigma}{n-p+\sigma}$. For the proof of removability of a singularity of the solution $u(x)$ it is sufficient to substitute in (5) $\varphi(x)=\psi(x)\,\eta_r(x)$ where $\psi(x)\in W^{1,p}(B)\cap L^q(B)$ and is equal to zero near ∂B, $\eta_r(x)$ is the same function that was used in the proof of the Theorem 1. Then passing to the limit if $r\to 0$ we obtain that $u(x)$ is the solution of (1) in B. This is the end of the proof of the Theorem 3.

1. *Kondratiev V.A., Landis E.M.* Qualitative theory of second-order linear partial differential equations. (Russian) Partial differential equations, 3 (Russian), 99–215, 220, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988.
2. *Kondratiev V.A., Landis E.M.* On the qualitative properties of the solutions of a nonlinear second order equation. Math. Sbornik 135, 346–360 (1988).
3. *Véron L.* Singularities of solutions of second order quasilinear equations. Pitman Research Notes in Math. Series, Harlow: Longman, vol.353, 1996.
4. *Heinonen J., Kilpelainen T., Martio O.* Nonlinear potential theory of degenerate elliptic equations. Oxford Clarendon Press, 1993.
5. *Skrypnik I.V.* Methods for Analysis of Nonlinear Elliptic Boundary Value Problems. AMS Transl., Providence, Rhode Island, vol.139, 1994.

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